MINIMAL BLOCKING SETS IN $PG(n,2)$ AND COVERING GROUPS BY SUBGROUPS

A. ABDOLLAHI*, M. J. ATAEI, A. MOHAMMADI HASSANABADI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ISFAHAN
ISFAHAN 81746-73441
IRAN.

Abstract. In this paper we prove that a set of points B of $PG(n,2)$ is a minimal blocking set if and only if $\langle B\rangle = PG(d,2)$ with d odd and B is a set of $d+2$ points of $PG(d,2)$ no $d+1$ of them in the same hyperplane. As a corollary to the latter result we show that if G is a finite 2-group and n is a positive integer, then G admits a $\mathfrak{C}_{n+1}$-cover if and only if n is even and $G \cong (C_2)^n$, where by a $\mathfrak{C}_m$-cover for a group H we mean a set $\mathcal{C}$ of size m of maximal subgroups of H whose set-theoretic union is the whole H and no proper subset of $\mathcal{C}$ has the latter property and the intersection of the maximal subgroups is core-free. Also for all $n < 10$ we find all pairs (m,p) ($m > 0$ an integer and p a prime number) for which there is a blocking set B of size n in $PG(m,p)$ such that $\langle B\rangle = PG(m,p)$.

1. Introduction and results

Let G be a group. A set $\mathcal{C}$ of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G. If the size of $\mathcal{C}$ is n, we call $\mathcal{C}$ an n-cover for the group G. A cover $\mathcal{C}$ for a group G is called irredundant if no proper subset of $\mathcal{C}$ is a cover for G. A cover $\mathcal{C}$ for a group G is called core-free if the intersection $D = \bigcap_{M\in\mathcal{C}} M$ of $\mathcal{C}$ is core-free in G, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of G. A cover $\mathcal{C}$ for a group G is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of G. A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group.

Let n be a positive integer. Denote by $PG(n,q)$ the n-dimensional projective space over the finite field $\mathbb{F}_q$ of order q. A blocking set in $PG(n,q)$ is a set of points that has nonempty intersection with every hyperplane of $PG(n,q)$. A blocking set that contains a line is called trivial. A blocking set is called minimal if none of its proper subsets are blocking sets. For a blocking set B in $PG(n,q)$ we denote by $d(B)$ the least positive integer d such that B is contained in a d-dimensional subspace of $PG(n,q)$. Thus $d(B)$ is equal to the (projective) dimension of the subspace spanned by B in $PG(n,q)$.

For further studies in the topic of blocking sets see Chapter 13 of the second edition of Hirschfeld's book [10] and also see [14].

The problem of covering a finite group with subgroups of a specified order has been studied in [11] where also bounds on the size of such covers was found. From Proposition 2.5 of [11] which is proved by a deep theorem due to Blokhuis [2], the following result easily follows. We will require this as an auxiliary tool later.

Theorem 1.1. (See Proposition 2.5 of [11]) *Let p be a prime and let G be a finite p-group with a maximal irredundant n-cover. Then either $n \geq \frac{3(p+1)}{2}$ or $n = p+1$.*

In section 2 we give relations between non-trivial minimal blocking sets of size n and $\mathfrak{C}_n$-groups. Also we give a complete characterization of minimal blocking sets in $PG(n,2)$.

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*Corresponding author: a.abdollahi@math.ui.ac.ir.

e-mail: mj.ataey@sci.ui.ac.ir, aamohaha@yahoo.com.

In order to characterize all $\mathfrak{C}_n$-groups, we first need to know the structure of an elementary abelian $\mathfrak{C}_n$-group. This is equivalent to find pairs (m, p) ($m \in \mathbb{N}$ and p is a prime number) for which $PG(m, p)$ contains a non-trivial minimal blocking set of size n (See Propositions 2.1, 2.2 and 2.6).

Nontrivial minimal blocking sets in $PG(2, p)$ of size $\frac{3(p+1)}{2}$ exist for all odd primes p. Indeed, an example is given by the projective triangle: the set consisting of the points $(0, 1, -s^2)$, $(1, -s^2, 0)$, $(-s^2, 0, 1)$ with $s \in \mathbb{F}_p$.

In [6], the smallest non-trivial blocking sets of $PG(n, 2)$ ($n \geq 3$) with respect to t-spaces ($1 \leq t \leq n-1$) are classified. A complete classification of minimal blocking sets seems to be impossible, however in this paper we determine all minimal blocking sets in $PG(n, 2)$, namely we prove

Theorem 1.2. *A set of points B of $PG(n, 2)$ is a minimal blocking set if and only if $\langle B \rangle = PG(d, 2)$ with d odd and B is a set of $d + 2$ points of $PG(d, 2)$ no $d + 1$ of them in the same hyperplane.*

For positive integers n and m, we denote the direct product of m copies of the cyclic group C_n of order n by $(C_n)^m$.

Using Theorem 1.2 we characterize all finite 2-groups having a $\mathfrak{C}_{n+1}$-cover as follows:

Theorem 1.3. *Let G be a finite 2-group and let n be a positive integer. Then G admits a $\mathfrak{C}_{n+1}$-cover if and only if n is even and $G \cong (C_2)^n$.*

Groups with a $\mathfrak{C}_n$-cover for $n = 3, 4, 5$ and 6 are characterized without appealing to the theory of blocking sets; see [13], [7], [4] and [1], respectively.

In section 3 we give some results on p-groups (p prime) satisfying the property $\mathfrak{C}_n$ for some positive integer n.

In section 4 we characterize elementary abelian $\mathfrak{C}_n$-groups for $n \in \{7, 8, 9\}$ as follows:

Theorem 1.4. *Let G be a $\mathfrak{C}_7$-group. Then G is a p-group for a prime number p if and only if $G \cong (C_2)^6$ or $(C_3)^4$.*

Theorem 1.5. *Let G be a $\mathfrak{C}_8$-group. Then G is a p-group for a prime number p if and only if $G \cong (C_3)^4$ or $(C_7)^2$.*

Theorem 1.6. *Let G be a $\mathfrak{C}_9$-group. Then G is a p-group for a prime number p if and only if $G \cong (C_2)^8$ or $(C_3)^5$ or $(C_5)^3$.*

In these characterizations we use Theorem 1.1 as well as some lemmas, the proof of which will be given in section 3.

We use Theorems 1.4, 1.5 and 1.6 to give certain non-trivial minimal blocking sets in $PG(3, 3)$ of sizes 7 and 8; and in $PG(4, 3)$ of size 9.

Theorem 1.7. *(a) Nontrivial minimal blocking sets of size 7 exist in $PG(3, 3)$.*
(b) Nontrivial minimal blocking sets of size 8 exist in $PG(3, 3)$.
(c) Nontrivial minimal blocking sets of size 9 exist in $PG(4, 3)$.

As a corollary to Theorems 1.4, 1.5 and 1.6 and some known results we give in a table all pairs (m, p) ($m > 0$ an integer and p a prime number) for which there is a blocking set B of size $n < 10$ in $PG(m, p)$ such that $d(B) = m$.

2. Relations between blocking sets and $\mathfrak{C}_n$-groups and characterization of minimal blocking sets in $PG(n, 2)$

As we mentioned in section 1, by a blocking set in $PG(n, q)$, we mean a blocking set with respect to hyperplanes in $PG(n, q)$.

Now we give some notations and definitions as needed in the sequel. We denote the product of n copies of $\mathbb{F}_q$ by $(\mathbb{F}_q)^n$. We note that $(\mathbb{F}_q)^n$ is a vector space of dimension n over $\mathbb{F}_q$. If $b = (b_1, \ldots, b_n) \in (\mathbb{F}_q)^n$, we denote by M_b the set of elements $x = (x_1, \ldots, x_n) \in (\mathbb{F}_q)^n$, such that $b \cdot x = \sum_{i=1}^n b_i x_i$ is equal to zero. Note that if $0 \neq b$, then M_b is an $(n-1)$-dimensional subspace of the vector space $(\mathbb{F}_q)^n$ and every $(n-1)$-dimensional subspace of $(\mathbb{F}_q)^n$ equals to M_b for some non-zero $b \in (\mathbb{F}_q)^n$. Since for every

$0 \neq \lambda \in \mathbb{F}_q$, $M_b = M_{\lambda b}$, $M_{\mathfrak{p}}$ is well-defined for every point $\mathfrak{p}$ of $PG(n-1,q)$, and $M_{\mathfrak{p}}$ may be considered as a hyperplane in $PG(n-1,q)$. We now give some results which clarify the relations between non-trivial minimal blocking sets of size n and $\mathfrak{C}_n$-covers for groups.
The following Propositions 2.1, 2.2 and 2.5 are well-known and their proofs are straightforward.

Proposition 2.1. *Let B be a set of points in $PG(n,q)$. Then B is a blocking set in $PG(n,q)$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is a $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Proposition 2.2. *Let B be a set of points in $PG(n,q)$. Then B is a minimal blocking set in $PG(n,q)$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is an irredundant $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Remark 2.3. Note that if q is prime, then the cover $\mathcal{C}$ in the statements of Propositions 2.1 and 2.2 is a maximal cover for $(\mathbb{F}_q)^{n+1}$.

Remark 2.4. It is easy to see that a (minimal) blocking set B with $d(B) = d$ in $PG(n,q)$ can be obtained from a (minimal) blocking set in $PG(d,q)$. So if we adopt an induction process on n to find all minimal blocking sets B in $PG(n,q)$, we must find only all those minimal blocking sets with $d(B) = n$.

Proposition 2.5. *Let B be a set of points in $PG(n,q)$. Then B is a blocking set with $d(B) = n$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is a core-free $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Proposition 2.6. *Let p be a prime number and n be a positive integer. Then a finite p-group G admits a $\mathfrak{C}_{n+1}$-cover if and only if $G \cong (C_p)^{m+1}$ for some positive integer m such that $PG(m,p)$ has a minimal blocking set B with $d(B) = m$ and $|B| = n+1$.*

Proof. Let G be a finite p-group admitting a $\mathfrak{C}_{n+1}$-cover. Then G has a maximal irredundant core-free $(n+1)$-cover, $\mathcal{C} = \{M_i \mid i = 1,\dots,n+1\}$ say. Since the Frattini subgroup $\Phi(G)$ of G is contained in M_i for every $i \in \{1,\dots,n+1\}$, $\Phi(G) \leq D_G = 1$, where D is the intersection of the cover $\mathcal{C}$. Hence $\Phi(G) = 1$ and so G is isomorphic to $(C_p)^{m+1}$ for some positive integer m. Now Propositions 2.2 and 2.5 and Remark 2.3 complete the proof. □

Let B be a set of points in $PG(n,q)$. Call any $|B| \times (n+1)$ matrix whose rows are generators of points of B a *blocking matrix* of B (regard a point in $PG(n,q)$ as a one dimensional subspace in $(\mathbb{F}_q)^{n+1}$).

Consider the following properties of a blocking matrix A of a set of points B in $PG(n,q)$

(a) The $|B| \times 1$ column matrix AX has at least one zero entry for every $(n+1) \times 1$ column matrix X with entries from $\mathbb{F}_q$.

(b) For each $i \in \{1,\dots,|B|\}$, there is an $(n+1) \times 1$ matrix X_i with entries from $\mathbb{F}_q$ such that the ith entry of AX_i is zero and all the others are non-zero.

Proposition 2.7. *Let B be a set of points in $PG(n,q)$ and let A be any blocking matrix of B. Then*

1. *B is a blocking set in $PG(n,q)$ with $d(B) = rank(A) - 1$ if and only if a blocking matrix of B satisfies the property (a).*
2. *B is a minimal blocking set in $PG(n,q)$ with $d(B) = rank(A) - 1$ if and only if a blocking matrix of B satisfies the properties (a) and (b).*

Proof. It follows from Propositions 2.2 and 2.5. □

In the following we apply a well-known idea which is used in coding theory to define an equivalence on linear codes (see e.g, pp. 50-51 in [9]).

Let A_1 and A_2 be two matrices of the same size with entries from $\mathbb{F}_q$. We say that A_1 is *equivalent* to A_2, if A_2 can be obtained from A_1 by a sequence of operations of the following types:

(C1) Permutation of the columns.
(C2) Multiplication of a column by a non-zero scalar from $\mathbb{F}_q$.
(C3) Addition of a scalar multiple of one column to another.
(R1) Permutation of the rows.
(R2) Multiplication of any row by a non-zero scalar.

Theorem 2.8. *Let B be a minimal blocking set in $PG(n,q)$ and let A be a blocking matrix of B. If A' is a matrix obtained from A by one of the operations (C1) to (R2), then (the points generated by) the rows of A' form a minimal blocking set B' in $PG(n,q)$ with $d(B)=d(B')$ and $|B|=|B'|$.*

Proof. Using Proposition 2.7 and noting that

$$AX = x_1A_1 + \cdots + x_{n+1}A_{n+1},$$

where $A_1,\ldots,A_{n+1}$ are columns of A and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix}$, the proof is straightforward. □

We say that two minimal blocking sets are *equivalent* if any blocking matrix of one of them is equivalent to any blocking matrix of the other.

Theorem 2.9. *Let A be any blocking matrix of a minimal blocking set B in $PG(n,q)$, let $k=|B|$ and $d=d(B)$. Then A is equivalent to a matrix of the form $\left[\begin{array}{c|c} I_{d+1} & \\ --- & K \\ L & \end{array}\right]$, where I_{d+1} is the $(d+1)\times(d+1)$ identity matrix, L is a $(k-d-1)\times(d+1)$ matrix and K is a $k\times(n-d)$ matrix. Also B can be obtained from a blocking set $\bar{B}$ with $d(\bar{B})=d$ in $PG(d,q)$ such that any blocking matrix of $\bar{B}$ is equivalent to a matrix of the form $\begin{bmatrix} I_{d+1} \\ L \end{bmatrix}$, where L is a $(k-d-1)\times(d+1)$ matrix.*

Proof. It is straightforward, see e.g., the proof of Theorem 5.5 in p. 51 of [9]. □

Proof of Theorem 1.2. We prove the following statement which is slightly more general than the statement of Theorem 1.2:
A minimal blocking set B with $d(B)=d$ in $PG(n,2)$ exists if and only if d is odd, $|B|=d+2$ and B can be obtained from a blocking set $\bar{B}$ with $d(\bar{B})=d$ in $PG(d,2)$ such that any blocking matrix of $\bar{B}$ is equivalent to a $(d+2)\times(d+1)$ matrix of the form

$$\begin{bmatrix} & & & I_{d+1} & & \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Let B be a minimal blocking set in $PG(n,2)$ with $d(B)=d$. By Theorem 2.9, B can be obtained from a minimal blocking set $\bar{B}$ in $PG(d,q)$ with $d(\bar{B})=d$ such that any blocking matrix of $\bar{B}$ is equivalent to a matrix $A' = \begin{bmatrix} I_{d+1} \\ L \end{bmatrix}$, where L is a $(k-d-1)\times(d+1)$ matrix, and $k=|B|=|\bar{B}|$. Let $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{d+1} \end{bmatrix}$ be an arbitrary row of L. Note that the column matrix X_i which satisfies Property (2) for A' in Proposition 2.7 is unique, for every $i\in\{1,\ldots,k\}$ and indeed X_i is the $(d+1)\times 1$ matrix in which the ith entry is zero and all other entries are 1. Thus $\mathbf{a}\cdot X_i \neq 0$ and so

$$a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_{d+1}=1 \text{ for all } i\in\{1,\ldots,d+1\}. \qquad (I)$$

On the other hand, by Proposition 2.7(1), the column matrix $A'\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ must have a zero entry. But all $(d+1)$ first entries are non-zero, so we have that

$$a_1+\cdots+a_{d+1}=0. \qquad (II)$$

Now it follows from (I) and (II) that $\mathbf{a} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$. Therefore L must have only one row which equals to $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$. Now equality (II) implies that $\underbrace{1+1+\cdots+1}_{d+1}=0$ from which it follows that d must be odd. This completes the proof. □

Proof of Theorem 1.3. It follows from Proposition 2.6 and Theorem 1.2. □

3. p-Groups with a $\mathfrak{C}_n$-cover

We shall need the following lemma in the sequel.

Lemma 3.1. (See Lemma 2.2 of [4]) *Let $\Gamma = \{A_i \ : \ 1 \leq i \leq m\}$ be an irredundant covering of a group G whose intersection of the members is D.*

(a) *If p is a prime, x a p-element of G and $|\{i : x \in A_i\}| = n$, then either $x \in D$ or $p \leq m - n$.*

(b) $\underset{j \neq i}{\cap} A_j = D$ *for all* $i \in \{1, 2, \ldots, m\}$.

(c) *If* $\underset{i \in S}{\cap} A_i = D$ *whenever* $|S| = n$, *then* $\left| \underset{i \in T}{\cap} A_i : D \right| \leq m - n + 1$ *whenever* $|T| = n - 1$.

(d) *If Γ is maximal and U is an abelian minimal normal subgroup of G. Then if $|\{i : U \subseteq A_i\}| = n$, either $U \subseteq D$ or $|U| \leq m - n$.*

We now prove some key lemmas.

Lemma 3.2. *Let G be a finite p-group having a $\mathfrak{C}_n$-cover $\{M_i \mid i = 1, \ldots, n\}$. Then*

(a) $p \leq n - 1$.

(b) *If s is the integer such that $1 \leq s \leq n-2$ and $p = n - s$, then $\bigcap_{i \in S} M_i = 1$ for every subset S of $\{1, 2, \ldots, n\}$ with $|S| \geq s + 1$.*

(c) *If $n = p + 1$, then $G \cong (C_p)^2$.*

Proof. Any blocking set B of $PG(d, q)$ has at least $q + 1$ points, and equality holds if and only if B is a line of $PG(d, q)$. This corresponds, for $q = p$ prime, to points (a) and (c) of Lemma 3.2 (see also [11, Proposition 2.5]). We give here a group-theoretic proof for the points (a) and (c).

(a) Let x be a p-element in G. Then by Lemma 3.1 (a), we have $p \leq n - m$, where $m = |\{i : x \in M_i\}|$. Therefore $p \leq n - 1$.

(b) Let $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = s + 1$; and let $N := \bigcap_{i \in S} M_i$. Then $N \trianglelefteq G$, since $M_i \trianglelefteq G$. Now suppose, for a contradiction, that $N \neq 1$. Since $G = (\bigcup_{i \in S} M_i) \bigcup (\bigcup_{j \notin S} M_j)$ and $|G : M_k| = p$ for every $k \in \{1, \ldots, n\}$, by Lemma 3.2 of [17], we have $p \leq n - s - 1$. This contradiction completes the proof of part (b).

(c) By Proposition 2.6, G is a finite elementary abelian p-group and by part (b), $M_1 \cap M_2 = 1$. Thus $|G| = |G : M_1 \cap M_2| = p^2$ and so $G \cong (C_p)^2$. □

Lemma 3.3. *Let $G = (C_p)^d$ ($d \geq 2$ and p is a prime number) and suppose that G has a $\mathfrak{C}_n$-cover $\{M_i \mid i = 1, \ldots, n\}$. Let $T \subseteq \{1, 2, \ldots, n\}$.*
(a) *If $|T| = n - p$, then $|\bigcap_{i \in T} M_i| = 1$ or p.*
(b) *If $|T| = 2$, then $|\bigcap_{i \in T} M_i| = p^{d-2}$.*
(c) *$\bigcap_{i \in T} M_i = 1$ for some T of size d.*
(d) *If $\bigcap_{i \in S} M_i = 1$ whenever $|S| = d$ then $p \leq |\bigcap_{i \in T} M_i| \leq n - d + 1$ whenever $|T| = d - 1$.*

Proof. (a) By Lemma 3.2(b) $\bigcap_{i \in K} M_i = 1$ for every subset K of $\{1, 2, \ldots, n\}$ such that $|K| = n - p + 1$. Now by Lemma 3.1 (c), $|\bigcap_{i \in T} M_i : \bigcap_{j \in K} M_j| \leq p$ and since G is a p-group, $|\bigcap_{i \in T} M_i| = 1$ or p.

(b) Since each M_j is a maximal subgroup of G, $|G : M_j| = p$. Therefore $|G : \bigcap_{i \in T} M_i| = p^2$, and so $|\bigcap_{i \in T} M_i| = p^{d-2}$.

(c) Suppose that $M_i = M_{b_i}$, where $b_i \in PG(d-1, p)$ for every $i \in \{1, \ldots, n\}$. Then by Proposition 2.6, $B = \{b_i \mid i = 1, \ldots, n\}$ is a minimal blocking set of size n in $PG(d-1, p)$ such that $d(B) = d - 1$. By Proposition 2.7, $d = rank(A)$, where A is any blocking matrix of B. Therefore there exists a subset $T \subseteq \{1, \ldots, n\}$ such that $|T| = d$ and $\{b_i \mid i \in T\}$ is linearly independent. This implies that $\bigcap_{i \in T} M_i = 1$, as required.

(d) Since $|G| = p^d$, $|\bigcap_{i \in T} M_i| \geq p$ for all T with $|T| = d - 1$. Now Lemma 3.1(c) completes the proof. □

4. p-groups having a $\mathfrak{C}_n$-cover for $n \in \{7, 8, 9\}$

In this section we characterize all p-groups having a $\mathfrak{C}_n$-cover for $n = 7, 8$ and 9. We denote by $[n]$ the set $\{1, \ldots, n\}$ and the set of all subsets of $[n]$ of size m will be denoted by $[n]^m$. We use the following results derived from the theory of blocking sets.

Remark 4.1. A minimal blocking set of $PG(2,q)$ has at most $q\sqrt{q}+1$ points [3, Theorem 1 (i)]. From this it follows that if $(C_p)^3$ has a $\mathfrak{C}_n$-cover then $n \leq p\sqrt{p}+1$.

Remark 4.2. A minimal blocking set of $PG(3,p)$ with $p>3$ prime of size at most $3(p+1)/2+1$ is contained in a plane [8, Theorem 1.4]. This implies the non-existence of a $\mathfrak{C}_9$-cover for $(C_5)^4$.

Remark 4.3. A minimal blocking set B of $PG(3,q)$ has at most q^2+1 points and equality holds if and only if B is an ovoid [3, Theorem 1 (ii)]. Also in [12] it has been proven that minimal blocking sets of size q^2 in $PG(3,q)$ do not exist. Therefore there is no $\mathfrak{C}_9$-cover for $(C_3)^4$.

Lemma 4.4. *Let G be a 3-group. Then G is a $\mathfrak{C}_7$-group, if and only if $G \cong (C_3)^4$.*

Proof. Suppose that G is a 3-group having a $\mathfrak{C}_7$-cover $\{M_i \mid i \in [7]\}$. By Proposition 2.6, G is an elementary abelian 3-group. By Lemma 3.2(b), $|G| \leq 3^5$.

Since an elementary abelian group of order 9 has only four maximal subgroups, we have $|G| \geq 27$. From Remark 4.1, it follows that $|G| \neq 27$.

Assume that $|G| = 3^4$ so that $G \cong (C_3)^4$. Now it is easy to check (e.g. by GAP [15]) that if $G = \langle a,b,c,d\rangle$, then the set

$$\mathcal{C} = \{\langle a,b,c\rangle, \langle a,c,d\rangle, \langle b,c,d\rangle, \langle a,b,d\rangle, \left\langle a,b,c^{-1}d\right\rangle, \left\langle a^{-1}b,c,d\right\rangle, \left\langle ad, a^{-1}c, ab\right\rangle\}$$

of maximal subgroups forms a $\mathfrak{C}_7$-cover for G.

Now let $|G| = 3^5$. Then by Lemma 3.3(b),

$$|M_i \cap M_j| = 27 \text{ for all distinct } i,j \in [7]. \tag{1}$$

Since $|G| = 3^5$, there is no subset $S \in [7]^3$ such that $|\bigcap_{i\in S} M_i| \leq 3$. Thus $|\bigcap_{i\in S} M_i| \in \{9,27\}$. Suppose, for a contradiction, that there exists $L \in [7]^3$ such that $|\bigcap_{i\in L} M_i| = 27 \quad (*)$. Let $L' \in [7]^2$ such that $L' \cap L = \varnothing$. Thus by Lemma 3.2(b), we have $|\bigcap_{i\in L\cup L'} M_i| = 1$. Now if $L'' \subset L$ such that $|L''| = 2$, then (1) and $(*)$ imply that $|\bigcap_{i\in L''\cup L'} M_i| = 1$. Since $|L''\cup L'| = 4$, it follows that $|G| \leq 3^4$, which is a contradiction. Therefore $|\bigcap_{i\in S} M_i| = 9$ for every $S \in [7]^3$ and so we can apply point (d) of Lemma 3.3 to get that $|\bigcap_{i\in T} M_i| = 3$ for every $T \in [7]^4$. Now it follows from Lemma 3.2(b) that $\bigcap_{i\in K} M_i = 1$ for all $K \subseteq [7]$ with $|K| \geq 5$. Now the inclusion-exclusion principle implies that $|\bigcup_{i=1}^7 M_i| = 225$, which is impossible. This completes the proof. □

Proof of Theorem 1.4. Let G be a p-group having a $\mathfrak{C}_7$-cover $\{M_i \mid i \in [7]\}$. By Proposition 2.6, G is an elementary abelian p-group. Now Theorem 1.1 implies that $p = 2$ or 3. If $p = 2$, then it follows from Theorem 1.3 that $G \cong (C_2)^6$. If $p = 3$, then Lemma 4.4 implies that $G \cong (C_3)^4$, and the proof is complete. □

Proof of Theorem 1.7(a). Consider the $\mathfrak{C}_7$-cover $\mathcal{C}$ for $(C_3)^4$ obtained in Lemma 4.4 and the set

$$B = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,0,0),(0,0,1,1),(1,-1,1,-1)\}$$

in $(F_3)^4$. It is easy to check (e.g. by GAP [15]) that $\mathcal{C} = \{M_b \mid b \in B\}$. Now Propositions 2.1 and 2.2 imply that B is a minimal blocking set of size 7 in $PG(3,3)$. □

Lemma 4.5. *The group $(C_3)^5$ has no $\mathfrak{C}_8$-cover.*

Proof. Suppose, for a contradiction, that $(C_3)^5$ has a $\mathfrak{C}_8$-cover $\{K_1,\dots,K_8\}$, where $K_i = M_{b_i}$, $b_i \in (\mathbb{F}_3)^5$. Then by Propositions 2.2 and 2.5, $B = \{b_i \mid i = 1,\dots,8\}$ is a minimal blocking set for $PG(4,3)$ with $d(B) = 4$. Now it follows from Theorem 2.9 that a blocking matrix of B is equivalent to a matrix as follows

$$A = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix}^T,$$

where $\mathbf{e}_i$ is the vector in $(\mathbb{F}_3)^5$ whose i-th entry is 1 and the others are zero. Since the matrix

$$A\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$$

must have at least one zero entry,

$$\mathbf{x}_i\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T = [0], \tag{$*$}$$

for some $i \in \{1,2,3\}$. By permuting the rows $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, if necessary, we may assume that $i = 1$. Now $(*)$ implies that the sum of entries of $\mathbf{x}_1$ is zero. It follows that there are, up to column permutations, only the following 4 vectors in $(\mathbb{F}_3)^5$ with the latter property:

$$[-1,1,1,1,1], [0,1,-1,1,-1], [0,0,1,1,1], [0,0,0,1,-1]. \tag{\#}$$

These column permutations will not change the *set* of the top 5 rows of the blocking matrix A, as they are the rows of the 5×5 identity matrix. Therefore $PG(4,3)$ has a minimal blocking set

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{x}, \mathbf{y}, \mathbf{z}\},$$

where $\mathbf{x}$ is one of the vectors in $(\#)$ and $\mathbf{y}, \mathbf{z} \in (\mathbb{F}_3)^5$. Now Propositions 2.2 and 2.5 imply that the maximal subgroups

$$M_1 = M_{\mathbf{e}_1}, M_2 = M_{\mathbf{e}_2}, M_3 = M_{\mathbf{e}_3}, M_4 = M_{\mathbf{e}_4}, M_5 = M_{\mathbf{e}_5}, M_6 = M_{\mathbf{x}}, M_7 = M_{\mathbf{y}}, M_8 = M_{\mathbf{z}} \tag{$**$}$$

forms a $\mathfrak{C}_8$-cover of $(C_3)^5$. It is not hard to show (e.g., by GAP [15]) that for every choice of the vector $\mathbf{x}$ from $(\#)$ and for all non-zero vectors $\mathbf{y}, \mathbf{z} \in (\mathbb{F}_3)^5$, $(**)$ is not an irredundant cover, a contradiction. This completes the proof.

We give here the proof of the latter claim when $\mathbf{x} = [-1,1,1,1,1]$. Firstly we determine the vectors $\mathbf{e}_1, \dots, \mathbf{e}_5$ and the group $(C_3)^5$ in GAP as the following permutations and the group:

```
a:=(1,2,3);;b:=(4,5,6);;c:=(7,8,9);;d:=(10,11,12);;
e:=(13,14,15);; C35:=Group(a,b,c,d,e);;
```

This means that we have considered the permutations `a,b,c,d,e` instead of the vectors $\mathbf{e}_1, \dots, \mathbf{e}_5$, respectively and so, for example, the vector $\mathbf{x}$ is corresponded to the permutation `a^-1*b*c*d*e` (written in GAP command form). Now the maximal subgroups $M_1, \dots, M_6$ are as follows in GAP:

```
M1:=Group(b,c,d,e);;M2:=Group(a,c,d,e);;
M3:=Group(a,b,d,e);;M4:=Group(a,b,c,e);;M5:=Group(a,b,c,d);;
M6:=Group(a*b,b*c^-1,c*d^-1,d*e^-1);;
```

We now produce all unordered pairs $\{M_7, M_8\}$ of maximal subgroups of $(C_3)^5$ such that

$$\{M_1, \dots, M_6, M_7, M_8\}$$

is an 8-cover of $(C_3)^5$.

```
T:=MaximalSubgroups(C35);; D:=Difference(T,[M1,M2,M3,M4,M5,M]);;
C:=Combinations(D,2);; K:=Union(M1,M2,M3,M4,M5,M);;
F:=Filtered(C,i->Size(Union(K,Union(i)))=3^5);;
B:=List(F,i->Union([M1,M2,M3,M4,M5,M],i));;
```

Therefore the set `B` contains all 8-covers of $(C_3)^8$ which contain $M_1, \dots, M_6$. It remains to check whether there is an *irredundant* cover of B or not. The following program collect all irredundant covers from `B` into the set `R` (if there is any).

```
R:=[ ]; for i in [1..Size(B)] do Q:=Combinations(B[i],7); if (3^5
in List(Q,i->Size(Union(i))))=false then Add(R,B[i]); fi; od;
```

Finally we see that the set `R` is empty for this choice of the vector $\mathbf{x}$. Similarly, for the other selections, we get that `R` is empty. This proves the claim. $\square$

Lemma 4.6. *Let G be a 3-group. Then G is a $\mathfrak{C}_8$-group, if and only if $G \cong (C_3)^4$.*

Proof. Suppose that G is a 3-group having a $\mathfrak{C}_8$-cover $\{M_i \mid i \in [8]\}$. Proposition 2.6 implies that G is an elementary abelian 3-group. By Lemma 3.3(b)

$$|G : M_i \cap M_j| = 9 \text{ for all distinct } i, j \in [8] \qquad (I)$$

and Lemma 3.2(b) implies that

$$\text{for every subset } T \subseteq [8] \text{ with } |T| \geq 6, \ \bigcap_{i \in T} M_i = 1. \qquad (II)$$

It follows that $|G| \leq 3^6$. Since an elementary abelian group of order 9 has only four maximal subgroups, we have $|G| \geq 27$ and it follows from Remark 4.1 that $|G| \neq 27$.

Assume that $|G| = 3^4$ so that $G \cong (C_3)^4$. Now it is easy to check (e.g. by GAP [15]) that if $G = \langle a, b, c, d \rangle$, then the set

$$\mathcal{D} = \{\langle a, b, c\rangle, \langle a, c, d\rangle, \langle b, c, d\rangle, \langle a, b, d\rangle, \langle a, b, c^{-1}d\rangle, \langle a^{-1}b, c, d\rangle, \langle d, ac, b\rangle, \langle ad, c, a^2b\rangle\}$$

of maximal subgroups forms a $\mathfrak{C}_8$-cover for G.

It follows from Lemma 4.5 that $|G| \neq 3^5$.

Now let $|G| = 3^6$. Then (I) implies that for every $K \in [8]^3$ we have $|\bigcap_{i \in K} M_i| = 27$ or 81.

We now prove that

$$|\bigcap_{i \in K} M_i| = 27 \text{ for all } K \in [8]^3. \qquad (III)$$

Suppose, for a contradiction, that there exists $L \in [8]^3$ such that

$$|\bigcap_{i \in L} M_i| = 81. \qquad (*)$$

Let $L' \in [8]^3$ such that $L' \cap L = \varnothing$. Thus by (II), we have $|\bigcap_{i \in L \cup L'} M_i| = 1$. Now if $L'' \subset L$ such that $|L''| = 2$, then (I) and $(*)$ imply that $|\bigcap_{i \in L'' \cup L'} M_i| = 1$. Since $|L'' \cup L'| = 5$, it follows that $|G| \leq 3^5$, which is a contradiction.

Now (III) yields that for every $W \in [8]^4$, we have $|\bigcap_{i \in W} M_i| = 9$ or 27. By a similar argument as in the latter paragraph, one can prove that

$$|\bigcap_{i \in W} M_i| = 9 \text{ for all } W \in [8]^4. \qquad (IV)$$

Since $G \cong (C_3)^6$ and (II) holds, we can apply Lemma 3.3(d) for the cover and so

$$|\bigcap_{i \in T} M_i| = 3 \text{ for all } T \in [8]^5. \qquad (V)$$

Thus since $G = \bigcup_{i=1}^{8} M_i$, it follows from the inclusion-exclusion principle and the relations (I), (II), (III), (IV), (V) that $|G| = 705$, which is impossible. This completes the result. □

Proof of Theorem 1.5. Let G be a p-group having a $\mathfrak{C}_8$-cover $\{M_i \mid i \in [8]\}$. By Proposition 2.6, G is an elementary abelian p-group. Now Theorems 1.1 and 1.3 imply that $p = 3$ or $p = 7$. If $p = 3$, then by Lemma 4.6, we conclude that $G \cong (C_3)^4$. If $p = 7$, then Lemma 3.2(c) yields that $G \cong (C_7)^2$.

For the converse if $G \cong (C_7)^2$, then G is a $\mathfrak{C}_8$-group, since it has exactly 8 maximal subgroups. If $G \cong (C_3)^4$, then Lemma 4.6 completes the proof. □

Proof of Theorem 1.7(b). Consider the $\mathfrak{C}_8$-cover $\mathcal{D}$ for $(C_3)^4$ obtained in Lemma 4.6 and the set

$$B = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (0,0,1,1), (2,0,1,0), (1,1,0,2)\}.$$

It is easy to check (e.g. by GAP [15]) that $\mathcal{D} = \{M_b \mid b \in B\}$. Now Propositions 2.1 and 2.2 imply that B is a minimal blocking set of size 8 in $PG(3,3)$. □

Remark 4.7. The examples of blocking sets in Theorem 1.7 (a) and (b) are particular examples of the following general families. Example (a) of Theorem 1.7 belongs to a family of minimal blocking sets of $PG(3,q)$ of size $2q+1$ constructed in [16, Examples 1] (see also [8, p. 171 Examples (d)]). Also, any example of this family, when $q=p$ is prime, provides a $\mathfrak{C}_{2p+1}$-cover of $(C_p)^4$. In the same paper other families of blocking sets of $PG(3,q)$ are presented that can be used to obtain other examples of covers of $(C_p)^4$. Whereas in [8, Corollary 3.6 (b)] are described examples of minimal blocking sets generating the whole space in $PG(d,p)$ for any p prime and any $d \geq 3$. Example (b) of Theorem 1.7 seems to belong to a family of minimal blocking sets of $PG(3,q)$ (q odd) of size $2q+2$ constructed in [8, Theorem 3.1 and Remarks (c)]. Also, any example of this family, when $q=p$ is an odd prime, produces a $\mathfrak{C}_{2p+2}$-cover of $(C_p)^4$.

Lemma 4.8. *Let G be a 5-group. Then G is a $\mathfrak{C}_9$-group if and only if $G \cong (C_5)^3$.*

Proof. Suppose that G is a 5-group. By Proposition 2.6, G is an elementary abelian 5-group. By Lemma 3.3(b)

$$|G : M_i \cap M_j| = 25 \text{ for all distinct } i,j \in [9]. \tag{1}$$

Now Lemma 3.2(b) implies that

$$\text{for every } \; T \subseteq [9] \text{ such that } |T| \geq 5, \; |\bigcap_{i\in T} M_i| = 1. \tag{2}$$

Therefore $|G| \leq 5^5$. Also by Lemma 3.3(a)

$$|\bigcap_{i\in W} M_i| \in \{1,5\} \;\text{ for all } W \in [9]^4. \tag{3}$$

Since an elementary abelian group of order 25 has only six maximal subgroups, we have $|G| \geq 5^3$.

Assume that $|G| = 5^3$ so that $G \cong (C_5)^3$. As the projective triangle in $PG(2,5)$ is a minimal blocking set of size 9, Proposition 2.6 implies that $(C_5)^3$ is a $\mathfrak{C}_9$-group. In fact if $G = \langle a,b,c\rangle$, then the set

$$\begin{gathered}\{\, \langle a,b\rangle, \langle a,c\rangle, \langle b,c\rangle, \langle b^3c,a\rangle, \langle a^3c,a^4b\rangle, \\ \langle a^2c,ab\rangle, \langle b^2c,a\rangle, \langle a^3c,ab\rangle, \langle a^2c,a^4b\rangle \,\}.\end{gathered}$$

of maximal subgroups forms a $\mathfrak{C}_9$-cover for G.

It follows from Remark 4.2 that $|G| \neq 5^4$.

Now if $|G| = 5^5$, then since (2) holds, Lemma 3.3(d) implies that

$$\text{for every subset } W \in [9]^4, \;\; |\bigcap_{i\in W} M_i| = 5. \tag{4}$$

Since $|G| = 5^5$, it follows from (1) that

$$|\bigcap_{i\in K} M_i| = 5^3 \;\text{ for all }\; K \in [9]^2, \tag{5}$$

and so $|\bigcap_{i\in K} M_i| \in \{25,125\}$ for every $K \in [9]^3$.

We prove that

$$|\bigcap_{i\in K} M_i| = 25 \;\text{ for all } K \in [9]^3. \tag{6}$$

Since otherwise there exists $L \in [9]^3$ such that $|\bigcap_{i\in L} M_i| = 125$. Let $L' \in [9]^2$ such that $L' \cap L = \varnothing$. Then (5) and (2) imply that

$$\bigcap_{i\in L''\cup L'} M_i = \bigcap_{i\in L\cup L'} M_i = 1$$

for every $L'' \subset L$ of size 2. This implies that $|G| \leq 5^4$, which is a contradiction.

Now using (1), (2), (4), (5) and (6), it follows from the inclusion-exclusion principle that $|G| = 2665$, which is a contradiction. □

Lemma 4.9. *The group $(C_3)^6$ is not a $\mathfrak{C}_9$-group.*

Proof. By a similar argument as in Lemma 4.5, one may prove the lemma. Note that here the vectors in $(\mathbb{F}_3)^6$ whose sum of its entries is zero, up to column permutations, are the following:

$$[1,1,1,1,1,1], [-1,1,-1,1,-1,1], [0,-1,1,1,1,1], [0,0,1,-1,1,-1], [0,0,0,1,1,1], [0,0,0,0,1,-1]. \quad (*)$$

Therefore by a similar argument as in Lemma 4.5, we complete the proof by proving that there is no minimal blocking set of size 9 for $PG(5,3)$ containing the standard basis vectors $\mathbf{e}_1,\dots,\mathbf{e}_6$ and one of the above vectors in $(*)$. □

Lemma 4.10. *Let G be a 3-group. Then G is a $\mathfrak{C}_9$-group, if and only if $G \cong (C_3)^5$.*

Proof. Suppose that G is a 3-group having a $\mathfrak{C}_9$-cover. Therefore by Lemma 3.3

$$|G : M_i| = 3 \text{ and } |G : M_i \cap M_j| = 9 \text{ for all distinct } i,j \in [9]. \quad (i)$$

By Proposition 2.6, G is an elementary abelian 3-group. By Lemma 3.2(b) we have

$$\text{for all } T \subseteq [9] \text{ such that } |T| \geq 7,\ |\bigcap_{i\in T} M_i| = 1. \quad (ii)$$

Therefore $|G| = 3^d$, where $d \leq 7$. Since an elementary abelian group of order 9 has only four maximal subgroups, $|G| \geq 27$ and it follows from Remarks 4.1 and 4.3 that $|G| \notin \{27, 81\}$. Therefore $5 \leq d \leq 7$.

Assume that $|G| = 3^5$ so that $G \cong (C_3)^5$. Now it is easy to see (e.g. by GAP [15]) that if $G = \langle a,b,c,d,e\rangle$, then the set

$$\mathcal{F} = \big\{ \langle a,b,c,d\rangle, \langle a,b,c,e\rangle, \langle a,b,d,e\rangle, \langle a,c,d,e\rangle, \langle b,c,d,e\rangle, \\ \left\langle a^{-1}c,b,d,e\right\rangle, \left\langle b^{-1}c,a,d,e\right\rangle, \langle de,c,b,a\rangle, \left\langle a^{-1}e,a^{-1}d,c,ab\right\rangle \big\}$$

of maximal subgroups forms a $\mathfrak{C}_9$-cover for G.

Now assume that $|G| = 3^7$. Then it follows from (i) that for every $K \in [9]^3$, we have $|\bigcap_{i\in K} M_i| = 3^4$ or 3^5. We now prove that

$$|\bigcap_{i\in K} M_i| = 3^4 \text{ for all } K \in [9]^3. \quad (*)$$

Suppose, for a contradiction, that there exists $L \in [9]^3$ such that $|\bigcap_{i\in L} M_i| = 3^5$. Let $L' \in [9]^4$ such that $L' \cap L = \varnothing$. Then it follows from (i) and (ii) that $|\bigcap_{i\in L''\cup L'} M_i| = 1$ for every $L'' \subset L$ of size 2. This implies that $|G| \leq 3^6$, a contradiction.

Now since $|G| = 3^7$, it follows from $(*)$ that for every $V \in [9]^4$, we have $|\bigcap_{i\in V} M_i| = 3^4$ or 3^3. By a similar argument as in the previous paragraph one can prove that for all $V \in [9]^4$, all $W \in [9]^5$ and all $T \in [9]^6$

$$|\bigcap_{i\in V} M_i| = 27, \qquad |\bigcap_{i\in V} M_i| = 9 \quad \text{and} \quad |\bigcap_{i\in T} M_i| = 3. \quad (**)$$

Now using $(i)-(ii)$ and $(*)-(**)$, it follows from the inclusion-exclusion principle that $|G| = 2125$, which is a contradiction. □

Proof of Theorem 1.6. Let G be a p-group having a $\mathfrak{C}_9$-cover $\{M_i \mid i = 1,\dots,9\}$. By Proposition 2.6, G is an elementary abelian p-group. By Theorem 1.1, $p \leq 5$. Now Lemmas 4.8, 4.10 and Theorem 1.3 complete the proof. □

Proof of Theorem 1.7(c). Consider the $\mathfrak{C}_9$-cover $\mathcal{F}$ for $(C_3)^5$ obtained in Lemma 4.10 and the set

$$B = \{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0), \\ (0,0,0,0,1),(1,0,1,0,0),(0,1,1,0,0),(0,0,0,1,-1),(0,1,0,-1,-1)\}.$$

It is easy to check (e.g. by GAP [15]) that $\mathcal{D} = \{M_b \mid b \in B\}$. Now Propositions 2.1 and 2.2 imply that B is a minimal blocking set of size 9 in $PG(4,3)$. □

We end this paper by posing

Problem 4.11. Let n be a positive integer. Find all pairs (m, p) ($m > 0$ an integer and p a prime number) for which there is a blocking set B of size n in $PG(m, p)$ such that $d(B) = m$.

The following table contains the complete answers for $n < 10$ to Problem 4.11 which have been exerted from the results of the present paper together with some known ones:

n	3	4	5	6	7	8	9
(m, p)	(1,2)	(1,3)	(3,2)	(1,5),(2,3)	(5,2), (3,3)	(1,7), (3,3)	(7,2),(2,5),(4,3)
Reference	[13]	[7]	[4]	[1]	Theorem 1.4	Theorem 1.5	Theorem 1.6

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